

# The Tower of Hanoi problem on $\text{Path}_h$ graphs

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## Abstract

The generalized Tower of Hanoi problem with  $h \geq 4$  pegs is known to require a sub-exponentially fast growing number of moves in order to transfer a pile of  $n$  disks from one peg to another. In this paper we study the  $\text{Path}_h$  variant, where the pegs are placed along a line, and disks can be moved from a peg to its nearest neighbor(s) only.

Whereas in the simple variant there are  $h(h-1)/2$  possible bi-directional interconnections among pegs, here there are only  $h-1$  of them. Despite the significant reduction in the number of interconnections, the number of moves needed to transfer a pile of  $n$  disks between any two pegs also grows sub-exponentially as a function of  $n$ .

We study these graphs, identify sets of mutually recursive tasks, and obtain a relatively tight upper bound for the number of moves, depending on  $h, n$  and the source and destination pegs.

Keywords: *Tower of Hanoi, path graphs, analysis of algorithms*

## 1 Introduction

In the well-known Tower of Hanoi problem, proposed over a hundred years ago by Lucas [20], a player is given 3 pegs and a certain number  $n$  of disks of distinct sizes, and is required to transfer them from one peg to another. Initially all disks are stacked (composing a tower) on the first peg (the source) ordered monotonically by size, with the smallest at the top and the largest at the bottom. The goal is to transfer them to the third peg (the destination), moving only topmost disks, and never placing a disk on top of a smaller one. The well-known recursive algorithm that accomplishes this task requires  $2^n - 1$  steps, and is the unique optimal algorithm for the problem. The educational aspects of the Tower of Hanoi puzzle have been reinforced recently, by a series of papers

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by Minsker ([23, 24, 25]), composing variants for the sake of studying their combinatorial as well as algorithmic aspects.

Work on this problem still goes on, studying properties of solution instances, as well as variants of the original problem. Connections between Pascal's triangle, the Sierpiński gasket and the Tower of Hanoi are established in [16], and to some classical numbers in [18]. In [1] it is shown that, with a certain way of coding the moves, a string which represents an optimal solution is square-free. This line is extended in [2]. Another direction was concerned with various generalizations, such as having any initial and final configurations [14], assigning colors to disks (cf. [21] and [22] for recent papers on the subject), and relaxing the placement rule of disks by allowing a disk to be placed on top of a smaller one under prescribed conditions [9, 10, 11].

A natural extension of the original problem is obtained by adding pegs. One of the earliest versions is "The Reve's Puzzle" [12, pp. 1-2]. There it was presented in a limited form: 4 pegs and specified numbers of disks. The general setup of the problem, with any number  $h > 3$  of pegs and any number of disks, was suggested in [28], with solutions in [29] and [15], shown recently to be identical [17]. An analysis of the algorithm reveals, somewhat surprisingly, that the solution grows sub-exponentially, at the rate of  $\Theta(\sqrt{n}2^{\sqrt{2n}})$  for  $h = 4$  (cf. [30]). The lower bound issue was considered in [32] and [8], where it has been shown that the minimal number of moves grows roughly at the same rate.

An imposition of movement restrictions among pegs generates many variants, and calls for representing variants by digraphs, where a vertex designates a peg, and an arc represents the permission to move a disk in the appropriate direction. In [3, 13], the uni-directional cyclic 3-peg variant ( $\text{Cyclic}_3$ ) has been studied, and the average distance between the nodes – in [31]. In [27], the "three-in-a-row" arrangement ( $\text{Path}_3$ ) is discussed. A unified treatment of all 3-peg variants is given in [26]. The (uni-directional)  $\text{Cyclic}_4$  is discussed for the first time in [27], and [30] studies other 4-peg variants:  $\text{Star}_4$  and  $\text{Path}_4$ , presenting a sub-exponential algorithm for  $\text{Star}_4$ . The  $\text{Cyclic}_h$  for any number of pegs  $h \geq 4$  has been studied in [6] and proved to be exponential for any specified  $h$ . Identification of the longest task, for certain variants, has been resolved in [7].

The only requirement for the problem to be solved for any number of disks is that the variant is represented by a strongly-connected directed graph. An interesting line of

research has been taken in [19], [4], and [5], where non-strongly-connected graphs are being studied.

In this paper we study the  $\text{Path}_h$  variant, with a fixed number  $h \geq 4$  of pegs, whose complexity issue has been left open. We devise an efficient algorithm which moves a column of  $n$  disks between any pair of pegs, and supply an explicit subexponential upper bound on the number of moves, for each  $h$ .

Notations and definitions are given in Section 2, the main results in Section 3, the proof of the 4-peg case in Section 4 and that of the general case in Section 5.

## 2 Preliminaries

We study the  $\text{Path}_h$  (a.k.a. *h-in-a-row*) variant, with a fixed number  $h \geq 4$  of pegs. We denote the pegs of  $\text{Path}_h$ , from left to right, by  $1, \dots, h$ . Let the sizes of the disks be  $1, 2, \dots, n$ . For convenience, we identify the name of a disk with its size.

For the statements and algorithms of the paper, it is required to introduce the notion of a *block* — a set of disks of consecutive sizes. The minimum (respectively, maximum) size of a disk in a block  $B$  is denoted by  $B_{\min}$  (resp.,  $B_{\max}$ ), and the number  $B_{\max} - B_{\min} + 1$  of disks in  $B$  — by  $|B|$ . A block  $B$  is *lighter* than another block  $B'$  if  $B_{\max} < B'_{\min}$ .

A *configuration* is a legal distribution of the  $n$  disks among the  $h$  pegs. A *perfect configuration* is one in which all the disks reside on the same peg. Such a configuration is denoted by  $R_{h,i,n}$ , where  $h$  is the number of pegs,  $i$  the peg holding the disks, and  $n$  the number of disks.

For a sequence of moves  $M$ , henceforth *move-sequence*, we denote by  $M^{-1}$  the reverse move-sequence, comprising the moves that cause the reverse effect. That is, the order of the moves is reversed and each move of the original sequence is reversed. Clearly, if applying  $M$  to configuration  $C_1$  results in reaching configuration  $C_2$ , then applying  $M^{-1}$  to  $C_2$  results in configuration  $C_1$ . (Note that this is true if and only if the peg structure is a graph; for digraphs in general this is not true.)

A problem instance, henceforth a *task*, is given by a pair of configurations, an *initial* configuration  $C_1$  and a *final* configuration  $C_2$ , where we are required to move from  $C_1$  to  $C_2$  in a minimal number of moves. The task, as well as a minimal-length solution of it, is denoted by  $C_1 \rightarrow C_2$ , and the minimum number of moves needed to get from  $C_1$  to  $C_2$  is denoted by  $|C_1 \rightarrow C_2|$ .

In this paper we focus on *perfect tasks* — problem instances whose initial and final configurations are both perfect. The peg associated with the initial (respectively, final) configuration of a perfect task is naturally referred to as the *source* (resp., *destination*). Clearly, for any positive integers  $h, n$  and  $1 \leq i < j \leq h$ , we have  $|R_{h,i,n} \rightarrow R_{h,j,n}| = |R_{h,j,n} \rightarrow R_{h,i,n}|$ . We shall henceforth restrict our attention in the sequel to tasks in which the source peg is to the left (i.e. has a lower peg index) of the destination peg.

For  $n \geq 1$ , denote by  $\text{Path}(h, n)$  the minimal number of moves which suffices for transferring a block of size  $n$  between all pairs of perfect configurations in  $\text{Path}_h$ , namely,

$$\text{Path}(h, n) = \max_{1 \leq i < j \leq h} |R_{h,i,n} \rightarrow R_{h,j,n}|.$$

For a real number  $x$ , let  $\text{round}(x)$  be the integer closest to  $x$  (where  $\text{round}(x) = \lceil x \rceil$  for  $x = n + 0.5$ ). For a pair of positive integers  $p$  and  $q$ , with  $p < q$ , we denote the set  $\{p, \dots, q\}$  by  $[p, q]$ , and  $[1, \dots, q]$  by  $[q]$ . In what follows, we do not distinguish between a move-sequence and an algorithm generating it, if this does not lead to a misunderstanding.

### 3 Main results

The main question the paper addresses is: what is the complexity of  $\text{Path}(h, n)$ ? An upper bound is provided by

**Theorem 3.1**  $\text{Path}(h, n) \leq C_h n^{\alpha_h} \cdot 3^{\theta_h \cdot n^{\frac{1}{h-2}}}$ , for all  $h \geq 3$  and  $n$ , where:

$$\begin{aligned} \theta_h &= ((h-2)!)^{\frac{1}{h-2}}, \\ \alpha_h &= \frac{h-3}{h-2}, \\ C_h &= \frac{(h-2) \cdot \delta^{h-3}}{\theta_h}, \quad \left( \delta = \frac{11}{3^{2-(1/30)^{1/3}}} \right). \end{aligned}$$

In particular,  $\text{Path}(h, n)$  grows subexponentially as a function of  $n$  for  $h \geq 4$ .

Of course, as a lower bound for  $\text{Path}(h, n)$  one may use any lower bound for the number of moves required to move a tower of size  $n$  from one peg to another over the complete graph on  $h$  vertices. By [8], such a lower bound is given by  $2^{(1+o(1))(n(h-2)!)^{\frac{1}{h-2}}}$ , which is “not very far” from our upper bound for  $\text{Path}(h, n)$ .

The following theorem identifies the hardest perfect task for the particular case  $h = 4$ . It also provides a tighter upper bound for  $\text{Path}_4$  than the one given in Theorem 3.1.

**Theorem 3.2** For every  $n \geq 1$ :

- (a)  $|R_{4,i,n} \rightarrow R_{4,j,n}| < |R_{4,1,n} \rightarrow R_{4,4,n}|$  for  $1 \leq i < j \leq 4, (i, j) \neq (1, 4)$ . In particular,  $\text{Path}(4, n) = |R_{4,1,n} \rightarrow R_{4,4,n}|$ .
- (b)  $\text{Path}(4, n) < 1.6\sqrt{n}3^{\sqrt{2n}}$ .

## 4 Proof of Theorem 3.2

### 4.1 On the relation between various tasks in $\text{Path}_4$

We start with a result of some independent interest, which holds for general  $h$ .

**Lemma 4.1** *Let  $C$  be a configuration with  $n \geq 1$  disks, arranged arbitrarily on pegs  $1, \dots, h-2$ , with pegs  $h-1$  and  $h$  empty. Then:*

$$|C \rightarrow R_{h,h-2,n}| < |C \rightarrow R_{h,h,n}|, \quad |C \rightarrow R_{h,h-1,n}| < |C \rightarrow R_{h,h,n}|.$$

**Proof:** We detail the proof for  $|C \rightarrow R_{h,h-2,n}| < |C \rightarrow R_{h,h,n}|$ . The proof for the second inequality is similar.

The proof is by induction on  $n$ . The basis  $n = 1$  is trivial. Let  $n \geq 2$ , assume that the statement holds for up to  $n-1$  disks, and let  $C$  be a configuration as in the statement of the lemma and  $M$  a move-sequence transferring from  $C$  to  $R_{h,h,n}$ . Before the last move of disk  $n$  (to peg  $h$ ), a configuration  $C'$ , in which all  $n-1$  disks  $1, 2, \dots, n-1$  are distributed among pegs  $1, \dots, h-2$ , is reached. Let  $M'$  (respectively,  $M''$ ) be the subsequence of  $M$ , consisting of all moves that come before (resp., after) the last move of disk  $n$ . Notice that  $M''$  transfers from  $C'$  (considered as a configuration of  $n-1$  disks) to  $R_{h,h,n-1}$ . By the induction hypothesis, there exists a move-sequence  $M''_{h-2}$  that transfers from  $C'$  to  $R_{h,h-2,n-1}$ , which is strictly shorter than  $M''$ . Let  $M'_{h-2}$  be the move-sequence obtained from  $M'$  by omitting all moves disk  $n$  makes after reaching peg  $h-2$  for the first time. Concatenating  $M'_{h-2}$  with  $M''_{h-2}$ , we obtain a legal move-sequence, strictly shorter than  $M$ , transferring from  $C$  to  $R_{h,h-2,n}$ . The required result follows.

Due to symmetries, there are actually only four essentially distinct perfect tasks in  $\text{Path}_4$ :  $R_{4,1,n} \rightarrow R_{4,2,n}$ ,  $R_{4,1,n} \rightarrow R_{4,3,n}$ ,  $R_{4,1,n} \rightarrow R_{4,4,n}$ , and  $R_{4,2,n} \rightarrow R_{4,3,n}$ . By Lemma 4.1, taking  $h = 4$  and  $C$  to be various perfect configurations, we obtain for any  $n \geq 1$

- $|R_{4,1,n} \rightarrow R_{4,2,n}| < |R_{4,1,n} \rightarrow R_{4,4,n}|$ .
- $|R_{4,2,n} \rightarrow R_{4,3,n}| < |R_{4,2,n} \rightarrow R_{4,4,n}| = |R_{4,1,n} \rightarrow R_{4,3,n}| < |R_{4,1,n} \rightarrow R_{4,4,n}|$ .

These inequalities establish part (a) of Theorem 3.2.

In Table 1 we present the (distinct) numbers  $|R_{4,i,n} \rightarrow R_{4,j,n}|$  for  $1 \leq n \leq 11$ . The entries have been calculated by finding the distance between the vertices  $R_{4,i,n}$  and  $R_{4,j,n}$  in the graph of all configurations of  $n$  disks on  $\text{Path}_4$  using breadth-first search.

disks	tasks				
	$2 \rightarrow 3$	$1 \rightarrow 2$	$1 \rightarrow 3$	$1 \rightarrow 4$	$\frac{1 \rightarrow 4}{\sqrt{n}3^{\sqrt{2n}}}$
1	1	1	2	3	0.634
2	4	4	6	10	0.786
3	7	9	12	19	0.744
4	14	18	22	34	0.760
5	23	29	36	57	0.790
6	34	44	54	88	0.799
7	53	69	78	123	0.762
8	78	96	112	176	0.768
9	105	133	158	253	0.798
10	138	182	212	342	0.795
11	187	241	272	449	0.783

Table 1: The minimal numbers of moves for the 4 different perfect tasks in  $\text{Path}_4$

The table prompts

**Question 1** Is it the case that  $|R_{4,1,n} \rightarrow R_{4,2,n}| < |R_{4,1,n} \rightarrow R_{4,3,n}|$  and  $|R_{4,2,n} \rightarrow R_{4,3,n}| < |R_{4,1,n} \rightarrow R_{4,2,n}|$  for all  $n \geq 3$ ?

Both of these inequalities seem intuitively quite plausible.

## 4.2 Upper bound for $\text{Path}(4, n)$

In this subsection we present the algorithm `FourMove` for moving a block  $B$  of size  $n$  from peg 1 to peg 4 in  $\text{Path}_4$ , requiring no more than  $1.6\sqrt{n}3^{\sqrt{2n}}$  moves. By Theorem 3.2(a), this will imply Theorem 3.2(b). The description of `FourMove` is given in Algorithm 1.

Prior to its main stages, `FourMove` partitions  $B$  into three: a block containing the smallest disks, a block containing the larger ones, and a block containing a single disk – the largest one. These blocks are denoted  $B_s, B_l, \{B_{\max}\}$  respectively, with  $m = |B_l \cup \{B_{\max}\}|$ . Thus  $B_s = [B_{\min}, B_{\max} - m]$  and  $B_l = [B_{\max} - m + 1, B_{\max} - 1]$ . In the three principal stages that follow: Spread, Circular shift and Accumulate, the moves are done based on these blocks. In Spread,  $B_s$  is transferred to the farthest peg – number 4,

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**Algorithm 1** FourMove( $B$ )

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/*  $B$  is a block of disks, partitioned into  $B_s \cup B_l \cup \{B_{\max}\}$ , where  $B_s$  is the set of smallest */
/* disks,  $B_l$  — the larger,  $\{B_{\max}\}$  — the largest. ThreeMove( $B, s, d, a$ ) returns the shortest */
/* move-sequence for moving  $B$  from  $s$  to  $d$  using  $a$ , referring to the graph induced by these */
/* three vertices as Path3. A move of disk  $n$  from peg  $i$  to peg  $i + 1$  is denoted by  $t_{i,i+1,n}$ . */
/* The '*' denotes concatenation of move-sequences. */
 $M \leftarrow []$  /* an empty sequence */
if  $B \neq \emptyset$  then
     $n \leftarrow |B|$ 
     $m \leftarrow \text{round}(\sqrt{2n})$ 
     $B_s \leftarrow [B_{\min}, B_{\max} - m]$ 
     $B_l \leftarrow [B_{\max} - m + 1, B_{\max} - 1]$ 
     $M_s \leftarrow \text{FourMove}(B_s)$ 
    /* Spread: */
     $M \leftarrow M_s * \text{ThreeMove}(B_l, 1, 3, 2) * t_{1,2,n}$ 
    /* Circular shift: */
     $M \leftarrow M * M_s^{-1} * \text{ThreeMove}(B_l, 3, 4, 2) * t_{2,3,n} * \text{ThreeMove}(B_l, 4, 2, 3)$ 
    /* Accumulate: */
     $M \leftarrow M * t_{3,4,n} * \text{ThreeMove}(B_l, 2, 4, 3) * M_s$ 
end if
return  $M$ 

```

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$B_l$  to peg 3,  $\{B_{\max}\}$  to peg 2. In Accumulate, the opposite is done: these blocks are gathered on the destination peg. In-between the algorithm performs the Circular shift stage, whose role is to reverse the order of the blocks, so that it will be possible to perform the Accumulate stage. It is easy to verify that, as the execution of the algorithm terminates, all the blocks are legally gathered on the destination peg 4, as required.

The ThreeMove procedure in Algorithm 1 produces move-sequences for  $B_l$  using only three (consecutive) pegs, which is exactly as moving it in Path<sub>3</sub>. To this end, we use the algorithm of [27], which transfers a block in a minimal number of moves between any two pegs in Path<sub>3</sub>, requiring  $3^n - 1$  moves to transfer  $n$  disks between the two farthest pegs, and half that number of moves between neighboring pegs.

Denote by  $T(n)$  the number of moves required by FourMove for a block of size  $n$ , and define  $T(0) = 0$ . Each of the three recursive invocations of the algorithm FourMove with  $B_s$  requires  $T(n - m)$  moves. Observe that, for a positive integer  $n$ , we have  $1 \leq m = \text{round}(\sqrt{2n}) \leq n$ . Employing the abovementioned results regarding the number of moves required by ThreeMove, it is easy to see that the total number of moves required by  $B_s$  is  $3(3^{m-1} - 1) + \frac{1}{2}(3^{m-1} - 1)$ . Finally,  $\{B_{\max}\}$  performs 3 moves. Altogether, for  $n \geq 1$

we have:

$$\begin{aligned} T(n) &= 3 \cdot T(n-m) + \frac{7}{2} \cdot (3^{m-1} - 1) + 3 \\ &= 3 \cdot T(n-m) + \frac{7}{6} \cdot 3^m - \frac{1}{2}, \end{aligned} \tag{1}$$

where  $m = \text{round}(\sqrt{2n})$ .

Next, we prove by induction that  $T(n) < 1.6\sqrt{n} \cdot 3^{\sqrt{2n}}$ , implying the required result. For the induction basis, we note that the inequality has been verified manually for all values of  $n \leq 8$ . Let  $n \geq 9$ , and assume that the inequality holds when  $n$  is replaced by a smaller integer. To prove it for  $n$ , denote first  $\beta = m - \sqrt{2n}$ . (Clearly,  $-\frac{1}{2} < \beta < \frac{1}{2}$ .) Note that

$$\sqrt{1 - \frac{m}{n}} < 1 - \frac{1}{\sqrt{2n}} - \frac{2\beta + 1}{4n}, \tag{2}$$

which can be verified by squaring both sides of the inequality, noting that the right-hand side is positive for  $n \geq 2$ . Thus, by the induction hypothesis and (2),

$$\begin{aligned} T(n) &= 3 \cdot T(n-m) + \frac{7}{6} \cdot 3^m - \frac{1}{2} \\ &< 3 \cdot 1.6\sqrt{n-m} \cdot 3^{\sqrt{2(n-m)}} + \frac{7}{6} \cdot 3^m \\ &= 3 \cdot 1.6\sqrt{n}\sqrt{1 - \frac{m}{n}} \cdot 3^{\sqrt{2n}\sqrt{1 - \frac{m}{n}}} + \frac{7}{6} \cdot 3^m \\ &< 3 \cdot 1.6\sqrt{n} \left( 1 - \frac{1}{\sqrt{2n}} - \frac{2\beta + 1}{4n} \right) 3^{\sqrt{2n}\left(1 - \frac{1}{\sqrt{2n}} - \frac{2\beta + 1}{4n}\right)} + \frac{7}{6} \cdot 3^{\sqrt{2n} + \beta} \\ &= 3^{\sqrt{2n}} \left[ 1.6\sqrt{n} \left( 1 - \frac{1}{\sqrt{2n}} - \frac{2\beta + 1}{4n} \right) 3^{-\frac{2\beta + 1}{2\sqrt{2n}}} + \frac{7}{6} \cdot 3^\beta \right] \\ &= 3^{\sqrt{2n}} \left[ 1.6\sqrt{n} \left( 1 - \frac{1}{\sqrt{2n}} - \frac{2\beta + 1}{4n} \right) e^{-\frac{2\beta + 1}{2\sqrt{2n}} \ln 3} + \frac{7}{6} \cdot 3^\beta \right]. \end{aligned}$$

Notice that the term  $-\frac{2\beta + 1}{2\sqrt{2n}} \ln 3$  is negative for  $\beta > -\frac{1}{2}$ . Hence, using the fact that  $e^x \leq 1 + x + \frac{1}{2}x^2$  for  $x < 0$ , we obtain



$$\begin{aligned}
\frac{T(n)}{1.6 \cdot 3^{\sqrt{2}n}} &< \sqrt{n} \left( 1 - \frac{1}{\sqrt{2n}} - \frac{2\beta+1}{4n} \right) \left( 1 - \frac{2\beta+1}{2\sqrt{2}n} \ln 3 + \frac{(2\beta+1)^2 \ln^2 3}{16n} \right) + \frac{35}{48} \cdot 3^\beta \\
&= \left( \sqrt{n} - \frac{1}{\sqrt{2}} - \frac{(2\beta+1) \ln 3}{2\sqrt{2}} \right) \\
&\quad + \left( -\frac{2\beta+1}{4\sqrt{n}} + \frac{(2\beta+1) \ln 3}{4\sqrt{n}} + \frac{(2\beta+1)^2 \ln^2 3}{16\sqrt{n}} \right) \\
&\quad + \left( \frac{(2\beta+1)^2 \ln 3}{8\sqrt{2} \cdot n} - \frac{(2\beta+1)^2 \ln^2 3}{16\sqrt{2} \cdot n} - \frac{(2\beta+1)^3 \ln^2 3}{64n\sqrt{n}} \right) + \frac{35}{48} \cdot 3^\beta \\
&= \left( \sqrt{n} - \frac{1}{\sqrt{2}} - \frac{(2\beta+1) \ln 3}{2\sqrt{2}} \right) \\
&\quad + \left( -\frac{2\beta+1}{4\sqrt{n}} + \frac{(2\beta+1) \ln 3}{4\sqrt{n}} + \frac{(2\beta+1)^2 \ln^2 3}{16\sqrt{n}} + \frac{(2\beta+1)^2 \ln^2 3}{16\sqrt{2} \cdot n} \right) \\
&\quad + \left( \frac{(2\beta+1)^2 \ln 3}{8\sqrt{2} \cdot n} - \frac{(2\beta+1)^2 \ln^2 3}{8\sqrt{2} \cdot n} - \frac{(2\beta+1)^3 \ln^2 3}{64n\sqrt{n}} \right) + \frac{35}{48} \cdot 3^\beta.
\end{aligned}$$

Observe that

$$\frac{(2\beta+1)^2 \ln 3}{8\sqrt{2} \cdot n} - \frac{(2\beta+1)^2 \ln^2 3}{8\sqrt{2} \cdot n} - \frac{(2\beta+1)^3 \ln^2 3}{64 \cdot n\sqrt{n}} < 0$$

for  $\beta > -\frac{1}{2}$ , and therefore

$$\begin{aligned}
\frac{T(n)}{1.6 \cdot 3^{\sqrt{2}n}} &< \left( \sqrt{n} - \frac{1}{\sqrt{2}} - \frac{(2\beta+1) \ln 3}{2\sqrt{2}} \right) \\
&\quad + \left( -\frac{2\beta+1}{4\sqrt{n}} + \frac{(2\beta+1) \ln 3}{4\sqrt{n}} + \frac{(2\beta+1)^2 \ln^2 3}{16\sqrt{n}} + \frac{(2\beta+1)^2 \ln^2 3}{16\sqrt{2} \cdot n} \right) + \frac{35}{48} \cdot 3^\beta \\
&= \left( \sqrt{n} - \frac{1}{\sqrt{2}} - \frac{(2\beta+1) \ln 3}{2\sqrt{2}} \right) \\
&\quad + \frac{2\beta+1}{16\sqrt{n}} \left( 4(\ln 3 - 1) + (2\beta+1) \ln^2 3 \left( 1 + \frac{1}{\sqrt{2n}} \right) \right) + \frac{35}{48} \cdot 3^\beta.
\end{aligned}$$

Now for  $\beta > -\frac{1}{2}$  the expression

$$\frac{2\beta+1}{16\sqrt{n}} \left( 4(\ln 3 - 1) + (2\beta+1) \ln^2 3 \left( 1 + \frac{1}{\sqrt{2n}} \right) \right) \tag{3}$$

increases as a function of  $\beta$  and decreases as a function of  $n$ . Hence its maximal value in the range  $-\frac{1}{2} < \beta \leq \frac{1}{2}, n \geq 9$ , is obtained for  $\beta = \frac{1}{2}, n = 9$ . Since its value at that point is less than 0.15, we have

$$\frac{T(n)}{1.6 \cdot 3^{\sqrt{2n}}} < \sqrt{n} + \left( -\frac{1}{\sqrt{2}} - \frac{(2\beta + 1) \ln 3}{2\sqrt{2}} + 0.15 + \frac{35}{48} \cdot 3^\beta \right). \quad (4)$$

It is easy to verify that for  $-\frac{1}{2} < \beta \leq \frac{1}{2}$ ,

$$f(\beta) = -\frac{1}{\sqrt{2}} - \frac{(2\beta + 1) \ln 3}{2\sqrt{2}} + 0.15 + \frac{35}{48} \cdot 3^\beta \quad (5)$$

is maximized at  $\beta = \frac{1}{2}$ , and  $f(\frac{1}{2}) < 0$ . Consequently, the right-hand side of (4) is smaller than  $\sqrt{n}$ , and we are done.

#### 4.2.1 A better upper bound for $\text{Path}(4, n)$

We reduced the problem of upper bounding  $\text{Path}(4, n)$  to the problem of upper bounding the following recurrence formula, which might be of independent interest:

$$T(n) = \min_{1 \leq m \leq n} \left( 3 \cdot T(n - m) + \frac{7}{6} \cdot 3^m - \frac{1}{2} \right).$$

We obtained an upper bound of  $1.6 \cdot \sqrt{n} \cdot 3^{\sqrt{2n}}$  for this recurrence formula, which is tight up to the leading constant 1.6.

One way to decrease the constant 1.6 is to show, using a computer program, that a better upper bound holds for all values of  $n \leq n'$ , for some huge integer  $n'$ . This will serve as a significantly more elaborate induction basis than the one that we use above (i.e.,  $n \leq 8$ ), and consequently, it would suffice to prove the induction step for  $n > n'$  only. The maximum value of the function  $g(n, \beta)$  defined in (3) for a huge integer  $n > n'$  and  $-\frac{1}{2} < \beta \leq \frac{1}{2}$  is some tiny number  $\varepsilon = \varepsilon(n)$ , and so, substituting 0.15 with  $\varepsilon$  in (5) yields:

$$f'(\beta) = -\frac{1.6}{\sqrt{2}} - \frac{1.6 \cdot \ln 3 (2\beta + 1)}{2\sqrt{2}} + 1.6 \cdot \varepsilon + \frac{7}{6} \cdot 3^\beta,$$

with  $f(\beta) - f'(\beta) = 1.6(0.15 - \varepsilon) \approx 1.6 \cdot 0.15$ . The difference  $1.6(0.15 - \varepsilon)$ , between  $f(\beta)$  and  $f'(\beta)$ , for a sufficiently small  $\varepsilon$ , enables us to decrease the leading constant 1.6 to approach 1.365, yielding an upper bound that approaches  $1.365 \cdot \sqrt{n} \cdot 3^{\sqrt{2n}}$ .

A more involved method for decreasing the above constant is to choose another value for  $m$ . For technical convenience, we fixed  $m = \text{round}(\sqrt{2n})$ , but this choice of  $m$  is inherently suboptimal. By following the method outlined in the previous paragraph, and setting  $m = \lfloor \sqrt{2n} + (1 - \frac{1}{\ln 3}) \rfloor$ , one can achieve a constant that approaches 1.105, yielding an upper bound of approximately  $1.105 \cdot \sqrt{n} \cdot 3^{\sqrt{2n}}$ .

## 5 Proof of Theorem 3.1

The proof of Theorem 3.1 is organized as follows. Generally, we would like to show how one can move a column of  $n$  disks from any source peg  $s$  to any destination peg  $d$  such that the number of moves is bounded above as the theorem states. For simplicity, we start by presenting an algorithm for the case where  $s = 1, d = h$ . This will be done in Section 5.1. Then we present an algorithm for the general case (Section 5.2). We note that, in fact, the first algorithm does employ the second. An important point in both cases is a partitioning of the set of disks to blocks, which will be discussed in Section 5.3. Time analysis of the two algorithms will be provided in Section 5.4.

### 5.1 Moving disks between the farthestmost pegs

Here we present **FarthestMove** (Algorithm 2), designed to move a block  $B$  of  $n$  disks between the two farthest pegs in  $\text{Path}_h$ , where  $h \geq 3$ .

We partition  $B$  in some way to blocks  $B_1(h, B), B_2(h, B), \dots, B_{h-1}(h, B)$  of disks. Whenever  $h$  and  $B$  are implied by the context, we write  $B_i$  instead of  $B_i(h, B)$ . The block  $B_1$  consists of the smallest  $\tilde{n}_1$  disks  $1, 2, \dots, \tilde{n}_1$ , the block  $B_2$  — of the  $\tilde{n}_2$  next smallest disks  $\tilde{n}_1 + 1, \tilde{n}_1 + 2, \dots, \tilde{n}_1 + \tilde{n}_2$ , and so forth. Similarly to the shorthand used when denoting blocks, we may write  $\tilde{n}_i$  (with a possible superscript) instead of  $\tilde{n}_i(h, n)$ . For any  $i \in [h - 1]$ , let  $B(i) = \bigcup_{j=i}^{h-1} B_j$  and  $n(i) = |B(i)| = \sum_{j=i}^{h-1} \tilde{n}_j$ , where  $\tilde{n}_j = |B_j|$ . (Note that  $B(1) = B$  and  $n(1) = n$ .)

The determination of the sizes  $\tilde{n}_i$  is crucial for the number of moves the algorithm makes, and will be explained later. However, for the algorithm to work correctly, it is only required for  $B_{h-1}$  to consist of the single disk  $B_{\max}$  — the largest. The algorithm consists of three phases (see Figure 1 for an illustration):

- **Spread:** Move the  $h - 2$  ( $= d - s - 1$ ) first blocks  $B_1, \dots, B_{h-2}$  from the source peg  $s$  to pegs  $d, \dots, d - h + 3$ , respectively. It consists of  $h - 2$  iterations. At the  $j$ -th iteration,  $j \in [h - 2]$ , block  $B_j$  is (recursively) moved from  $s$  to  $d - j + 1$ , using the set  $[1, d - j + 1]$  of available pegs. (Note that the 1-disk block  $B_{h-1}$  has not been moved from  $s$  to  $s + 1$ . It is more convenient for us to view this move as the first move of the next stage.)
- **Reverse:** The role of this phase is to reverse the positions of the  $h - 1$  blocks on the  $h$  pegs, i.e., a block residing, at the beginning of this phase, on peg  $s + i - 1$

reaches, at the end of the phase, its reflected position — peg  $d - i + 1$ . The phase starts by moving the last block  $B_{h-1}$  from  $s$  to  $s + 1$ . Then,  $h - 2$  rounds are carried out, each of which brings the next larger block to its reflected position. The following highlights the way each round  $j$  achieves its goal:

- Before this round, blocks  $B_1, \dots, B_{j-1}$  are on pegs  $s, \dots, s + j - 2$ , respectively; peg  $s + j - 1$  is vacant; blocks  $B_j, \dots, B_{h-1}$  are on pegs  $d, \dots, s + j$ , respectively.
- Block  $B_j$  is moved from  $d$  to  $s + j - 1$ .
- Blocks  $B_{j+1}, \dots, B_{h-1}$  are each shifted one peg to the right.
- At the end of the round, blocks  $B_1, \dots, B_j$  are on pegs  $s, \dots, s + j - 1$ , respectively; peg  $s + j$  is vacant; blocks  $B_{j+1}, \dots, B_{h-1}$  are on pegs  $d, \dots, s + j + 1$ , respectively.

Thus, as a result of this phase, block  $B_{h-1}$  is moved from  $s$  to  $d$  and, for each  $j \in [h - 2]$ , block  $B_j$  is moved from peg  $d - j + 1$  to the reflected position, namely, peg  $s + j - 1$ .

- **Accumulate:** The role of this phase is symmetrical to that of **Spread**, i.e., to move the  $h - 2$  first blocks  $B_{h-2}, \dots, B_1$  from pegs  $d - 2, \dots, s$ , respectively, to  $d$ . Similarly, it consists of  $h - 2$  iterations, where at the  $j$ -th iteration block  $B_{h-1-j}$  is moved from  $s + h - 2 - j$  to  $d$  using the set  $[s + h - 2 - j, d]$  of available pegs.

It is easy to verify that, as the execution of the algorithm terminates, all the blocks are legally gathered on  $d$ , as required. The formal description of the algorithm **FarthestMove** is given in Algorithm 2.

## 5.2 Moving disks between any pegs

The general algorithm for moving a block of disks, between any two pegs  $s$  and  $d$ , in  $\text{Path}_h$ , is presented here. For convenience we assume that  $s < d$ . This does not effect the generality of the algorithm since, as was mentioned in Section 2, if  $M$  is a solution of  $R_{h,s,n} \rightarrow R_{h,d,n}$ , then  $M^{-1}$  is a solution of  $R_{h,d,n} \rightarrow R_{h,s,n}$ .

The issue of partitioning the disk set is handled exactly as it was done in **FarthestMove**. Algorithm **GeneralMove** consists of five phases: two spread phases, a phase in which the remainder disks are moved, and two accumulate phases. The set of available pegs is denoted by  $A$ , and its smallest and largest pegs by  $A_{\min}$  and  $A_{\max}$ , respectively.

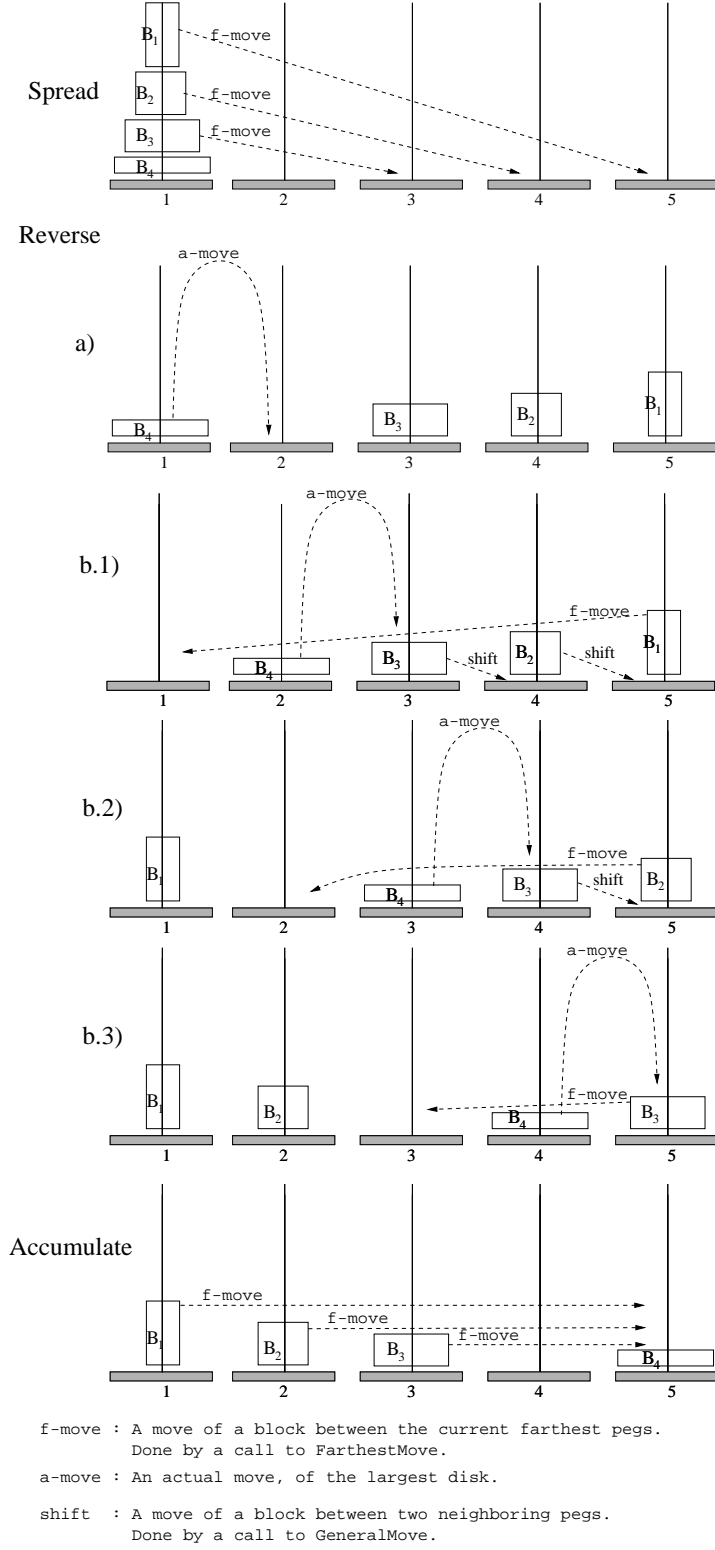


Figure 1: Main steps of FarthestMove for Path<sub>5</sub>, with  $B = \bigcup_{i=1}^4 B_i$ ,  $s = 1$  and  $d = 5$ .

---

**Algorithm 2** FarthestMove( $B, s, d$ )

---

```
/* The procedure moves a block  $B$  from the leftmost peg  $s$  to the rightmost peg  $d$ . Prior */
/* to its main stages, the block is partitioned into  $h - 1$  blocks which are treated as 'atomic */
/* units'. At the first of the main stages, these blocks are spread along the pegs; at the */
/* second – their order is reversed; at the third stage they are accumulated on the destination */
/* peg. The procedure requires that  $s < d$ . If  $d - s = 1$ , then  $|B| \leq 1$ . The '*' denotes */
/* concatenation of move-sequences. */
 $T \leftarrow []$  /* initializing the result sequence */
if  $B \neq \emptyset$  then
     $h \leftarrow d - s + 1$ 
     $(B_1, \dots, B_{h-1}) = \text{Partition}(h, B)$  /* Algorithm 5 below */

    /* Spread: */
    for  $j \leftarrow 1$  to  $h - 2$  do
        /* At each step, the next block moves to the farthest available peg. */
         $T \leftarrow T * \text{FarthestMove}(B_j, s, d - j + 1)$ 
    end for

    /* Reverse: */
     $T \leftarrow T * t_{B_{\max}, s, s+1}$  /* Moving the largest disk once a peg to the right. */
     $M \leftarrow []$  /* Initializing the temporary move-sequence. */
    for  $j \leftarrow 1$  to  $h - 2$  do
        /* Block  $B_j$  moves to the peg on which it will stay for the rest of this phase. */
         $M_f \leftarrow \text{FarthestMove}(B_j, s + j - 1, d)$ 
         $M \leftarrow M * M_f^{-1}$ 
        for  $i \leftarrow j + 1$  to  $h - 2$  do
            /* Each block whose index is higher than  $j$  is shifted one peg to the right. */
             $M \leftarrow M * \text{GeneralMove}(B_i, d + j - i, d + j + 1 - i, [s + j, d + j + 1 - i])$ 
        end for
         $T \leftarrow T * M * t_{B_{\max}, j+1, j+2}$  /* Moving the largest disk a peg to the right. */
    end for

    /* Accumulate: */
    for  $j \leftarrow 1$  to  $h - 2$  do
        /* At each step, the next block is gathered on the destination peg. */
         $T \leftarrow T * \text{FarthestMove}(B_{h-1-j}, s + h - 2 - j, d)$ 
    end for
end if
return  $T$ 
```

---

- **LeftSpread:** In this phase the  $s - A_{\min}$  first blocks  $B_1, \dots, B_{s-A_{\min}}$  are taken from peg  $s$  to pegs  $A_{\min}, \dots, s - 1$ , respectively. It consists of  $s - A_{\min}$  iterations. At the  $j$ -th iteration,  $1 \leq j \leq s - A_{\min}$ , block  $B_j$  is (recursively) moved from  $s$  to  $A_{\min} + j - 1$  using the set  $[A_{\min} + j - 1, A_{\max}]$  of available pegs.
- **RightSpread:** Here, the  $A_{\max} - d$  next blocks are taken, from peg  $s$  to pegs  $A_{\max}, \dots, d + 1$ , respectively. At each iteration  $j$ , where  $1 \leq j \leq A_{\max} - d$ , block  $B_{s-A_{\min}+j}$  is moved from  $s$  to  $A_{\max} - j + 1$ , using  $[s, A_{\max} - j + 1]$ . Since at each iteration the source and destination are at the opposite ends of the currently available set of free pegs, the move is done using algorithm **FarthestMove**.
- **MoveRemainder:** In this phase, the remaining  $B(d - s)$  blocks are moved from  $s$  to  $d$ . Since, as before, the source and destination are at the opposite sides of the set  $[s, d]$  of available pegs, this is done by algorithm **FarthestMove**.
- **LeftAccumulate:** The role of this phase is symmetrical to that of **RightSpread**, that is, move  $B_{s-A_{\min}+1}, \dots, B_{s+|A|-1-d}$  from  $A_{\max}, \dots, d + 1$  to  $d$ , respectively. It consists of  $A_{\max} - d$  iterations, where at iteration  $j$ , block  $B_{s+|A|-d-j}$  is moved from  $d + j$  to  $d$  using  $[s, d + j]$ . Unlike **RightSpread**, the moves made in this phase are not between the two farthest available pegs.
- **RightAccumulate:** This phase is symmetrical to **LeftSpread**, consisting of  $s - A_{\min}$  iterations where, at the  $j$ -th iteration,  $B_{s-j}$  is moved from peg  $s - j$  to peg  $d$ , using  $[s - j, A_{\max}]$ .

It is easy to verify that, as the algorithm terminates, all the blocks are legally gathered on the destination peg  $d$ , as required (see Figure 2 for an illustration). The correctness proof is omitted. The formal description of **GeneralMove** is given in Algorithm 3. Note that, if the source and destination pegs are at the opposite sides of  $A$ , then **GeneralMove** does the same as **FarthestMove**.

### 5.3 Partitioning the disks into blocks

In this section we discuss how to set the sizes of the blocks such that the number of moves will be relatively low. The general idea is to view the  $h - 1$  blocks as ‘atomic’ units, each occupying a single peg (except for when it is moved). During the process of moving a block  $B_i$  from one peg  $s$  to another peg  $d$ , the other blocks stay intact. Furthermore, the

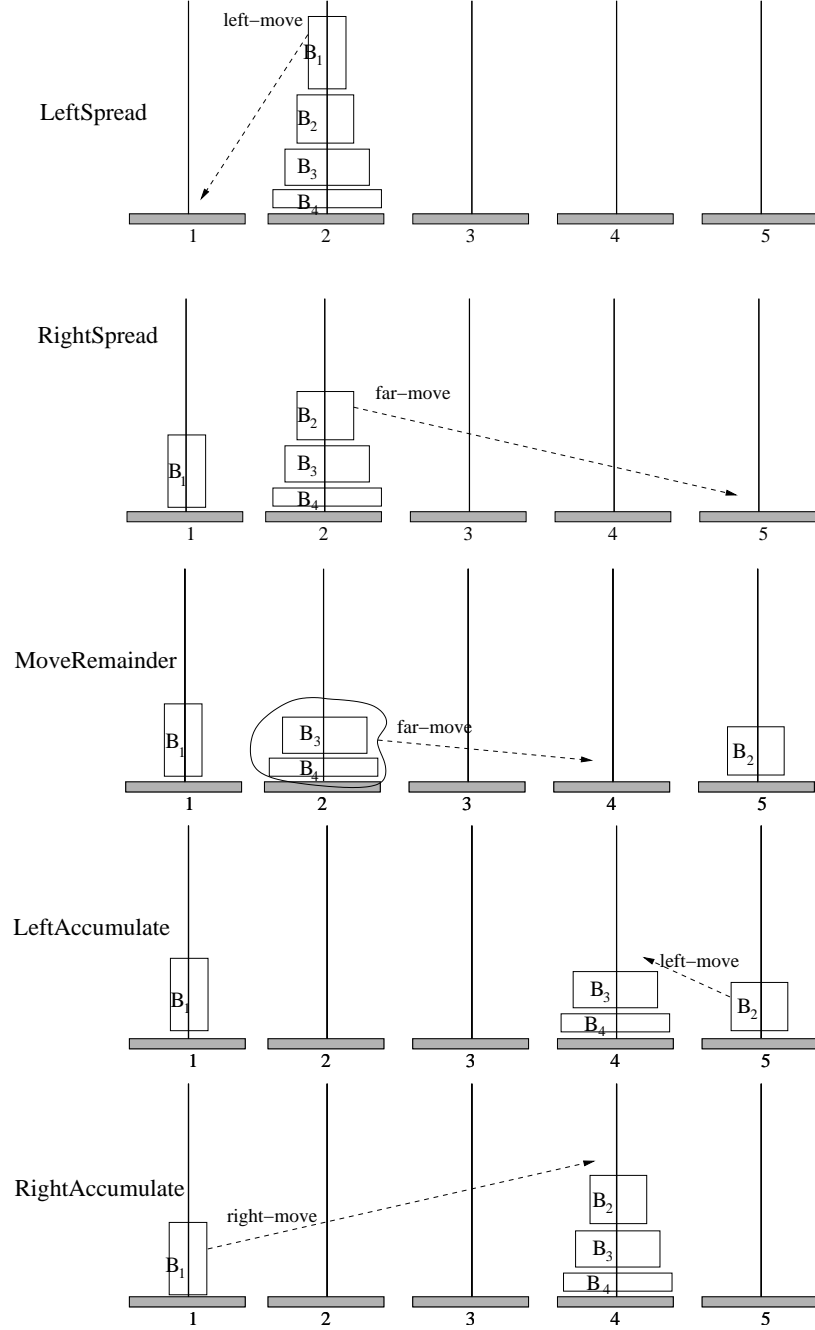


Figure 2: An illustration of the execution of the algorithm `GeneralMove` for `Path5`, with  $B = \bigcup_{i=1}^4 B_i$ ,  $s = 2$ ,  $d = 4$  and  $A = [5]$ .



---

**Algorithm 3** GeneralMove( $B, s, d, A$ )

---

```
/* The procedure moves a block  $B$  from any peg  $s$  to any other peg  $d$ . The partitioning is as in */
/* FarthestMove. Then some blocks are spread to the left of  $s$  and to the right of  $d$ ; next – the */
/* remaining blocks are moved; eventually, the smallest disks are accumulated on the destination */
/* peg. The procedure requires that  $s, d \in A, s < d$ . If  $d - s = 1$ , then  $|B| \leq 1$ . The '*' denotes */
/* concatenation of move-sequences. */
 $T \leftarrow []$  /* an empty sequence */
if  $B \neq \emptyset$  then
     $(B_1, \dots, B_{|A|-1}) = \text{Partition}(|A|, B)$  /* Algorithm 5 below */

    /* LeftSpread: */
    for  $j \leftarrow 1$  to  $s - A_{\min}$  do
        /* At each step, the next block moves to the farthest available peg on the left side. Since */
        /* the source is not necessarily at the rightmost available peg, this is done by GeneralMove. */
         $f_n \leftarrow A_{\min} + j - 1$ 
         $M \leftarrow \text{GeneralMove}(B_j, f_n, s, [f_n, A_{\max}])$ 
         $T \leftarrow T * M^{-1}$ 
    end for

    /* RightSpread: */
    for  $j \leftarrow 1$  to  $A_{\max} - d$  do
        /* At each step, the next block moves to the farthest available peg on the right side. */
        /* Since the source is at the leftmost available peg, this is done by FarthestMove. */
         $f_x \leftarrow A_{\max} - j + 1$ 
         $T \leftarrow T * \text{FarthestMove}(B_{s-A_{\min}+j}, s, f_x)$ 
    end for

    /* MoveRemainder: */
     $T \leftarrow T * \text{FarthestMove}(B(|A| - d + s), s, d)$ 

    /* LeftAccumulate: */
    for  $j \leftarrow 1$  to  $A_{\max} - d$  do
         $f_x \leftarrow d + j$ 
         $M \leftarrow \text{GeneralMove}(B_{|A|+s-d-j}, d, f_x, [s, f_x])$ 
         $T \leftarrow T * M^{-1}$ 
    end for

    /* RightAccumulate: */
    for  $j \leftarrow 1$  to  $s - A_{\min}$  do
         $f_n \leftarrow s - j$ 
         $T \leftarrow T * \text{GeneralMove}(B_{s-j}, f_n, d, [f_n, A_{\max}])$ 
    end for
end if
return  $T$ 
```

---

pegs used by disks from  $B_i$  during this process form an interval of contiguous integers, contained in the set of pegs available to this end, namely, the inclusion-wise maximal interval of pegs not occupied by any of the blocks  $B_1, \dots, B_{i-1}$ .

To move a block between pegs efficiently, all available pegs should usually be in use. More specifically, during the process of moving a sufficiently large block  $B_i$ , all of the available pegs are used. Furthermore, the algorithm allocates precisely  $h - i + 1$  pegs to this end. This suggests that, in order to perform efficiently, the sizes of the  $h - 1$  blocks should satisfy  $\tilde{n}_1 \geq \dots \geq \tilde{n}_{h-1} = 1$  (assuming  $n$  is sufficiently large).

### 5.3.1 The Partition procedure

In this section we present **Partition** — the procedure for partitioning a block  $B$  into the  $h - 1$  blocks  $(B_1, B_2, \dots, B_{h-1})$ . We start by presenting an auxiliary function **Remainder**, which, for each stage  $j$ , provides the total number of disks to be assigned to the latter blocks —  $(B_{j+1}, B_{j+2}, \dots, B_{h-1})$ . The definition of this function is given in Algorithm 4.

---

#### Algorithm 4 **Remainder**( $h, n$ )

---

```

/* Determines the size of the next set, by calculating the number of 'larger' disks. */
/* It is assumed that  $h \geq 3$  and  $n \geq 1$ . */
if  $n < h$  then
    return  $\max\{n - 1, 1\}$ 
else
    if  $h = 4$  then
        return  $\min\{n, \text{round}(\sqrt{2n})\}$ 
    else
         $\alpha \leftarrow \frac{h-3}{h-2}$ 
        return  $\min\left\{n, \left\lceil \frac{((h-2)!)^\alpha}{(h-3)!} n^\alpha \right\rceil\right\}$ 
    end if
end if

```

---

**Lemma 5.1** *For any integers  $h \geq 3$  and  $n \geq 1$ , we have  $1 \leq \text{Remainder}(h, n) \leq n$ . Furthermore,*

- *If  $h = 3$ , then  $\text{Remainder}(h, n) = 1$ .*
- *If  $h \geq 4$  and  $1 < n < h$ , then  $\text{Remainder}(h, n) = n - 1$ .*
- *If  $h \geq 4$  and  $n \geq h$ , then  $\text{Remainder}(h, n) \geq 2$ .*

The proof is straightforward.

---

**Algorithm 5** Partition( $h, B$ )

---

```
/* Returns a partition of the  $n$  disks into  $h - 1$  blocks of consecutive disks. */
/* It is assumed that  $h \geq 2$  and  $B$  is a non-empty block of disks. */
for  $j \leftarrow 1$  to  $h - 2$  do
     $n_j \leftarrow B_{\max} + 1 - B_{\min}$ 
     $m_j \leftarrow \text{Remainder}(h - j + 1, n_j)$ 
     $\tilde{n}_j \leftarrow n_j - m_j$ 
     $B_j \leftarrow [B_{\min}, B_{\min} + \tilde{n}_j - 1]$     /* if  $\tilde{n}_j = 0$ , then  $B_j = \emptyset$  */
     $B \leftarrow B - B_j$ 
end for
/*  $B$  is now a singleton, so that  $B_{h-1} = \{B_{\max}\}$  */
 $B_{h-1} \leftarrow B$ 
return  $(B_1, \dots, B_{h-1})$ 
```

---

The formal description of the procedure Partition is given in Algorithm 5.

We argue that Partition is well-defined. To prove this, it suffices to show that at each of the  $h - 2$  invocations of  $\text{Remainder}(h - j + 1, n_j)$ ,  $1 \leq j \leq h - 2$ , we have  $h - j + 1 \geq 3$  and  $n_j \geq 1$ . The first of these inequalities follows from the fact that  $j \leq h - 2$ . Now observe that  $n_1 = n \geq 1$ , and  $n_{j+1} = m_j = \text{Remainder}(h - j + 1, n_j)$ . Hence, by Lemma 5.1, a simple inductive argument yields

$$n_j \geq 1, \quad 1 \leq j \leq h - 2, \quad (6)$$

and we are done.

In the following lemma, whose proof is straightforward, we collect for later reference a few properties of the partition  $(B_1, \dots, B_{h-1})$ .

**Lemma 5.2** *The tuple  $(B_1, \dots, B_{h-1})$  is a partition of  $B$  into blocks, satisfying:*

- $B_{h-1} = \{B_{\max}\}$ .
- $|B_j| = \tilde{n}_j \leq n - 1$  for each  $j \in [h - 2]$ .
- Each non-empty block is lighter than all subsequent non-empty blocks in the partition.

It is easy to verify that, for a pair of indices  $i \in [h - 1]$  and  $j \in [h - i]$ ,

$$B_{j+i-1}(h, B) = B_j(h - i + 1, B(i)),$$

or, equivalently:

**Lemma 5.3** *For any integers  $1 \leq i \leq h - 1$  and  $1 \leq j \leq h - i$ :*

$$\tilde{n}_{j+i-1}(h, n) = \tilde{n}_j(h - i + 1, n(i)).$$

#### 5.4 FarthestMove versus GeneralMove

We assume without loss of generality that  $A = [1, h]$ . For any integers  $h \geq 3, n \geq 0, s$  and  $d$ , such that  $1 \leq s < d \leq h$ , we denote by  $G_{s \rightarrow d}(h, n)$  the number of moves required by GeneralMove to move a block of size  $n$  from peg  $s$  to peg  $d$  using  $A$ . Similarly, we denote by  $F(h, n)$  the number of moves required by FarthestMove to move such a block from peg 1 to peg  $h$ . (Note that  $F = G_{1 \rightarrow h}$ .)

It is easy to verify that, for  $h = 3$ , the algorithm GeneralMove works exactly as does the classical algorithm of [27]. In particular, it requires  $3^n - 1$  moves to transfer  $n$  disks between the two farthest pegs in  $\text{Path}_3$ , and  $\frac{3^n - 1}{2}$  moves to transfer them between neighboring pegs, yielding:

**Lemma 5.4** *For any non-negative integer  $n$ :*

$$G_{1 \rightarrow 2}(3, n) = \frac{1}{2}F(3, n).$$

##### 5.4.1 Initial steps in the analysis of GeneralMove

In this section we analyze the algorithm GeneralMove for moving a block  $B$  in  $\text{Path}_h$ ,  $h \geq 3$ , from peg  $s$  to peg  $d$ ,  $s < d$ , using the set  $A = [1, h]$  of available pegs. Let  $h' = s + h - d$ . (Note that  $h' < h$ .)

Consider an index  $j \in [s - 1]$ . At phase **LeftSpread**, a **left-move** of block  $B_j$  from peg  $s$  to peg  $j$  using  $h - j + 1$  available pegs is performed, requiring  $G_{j \rightarrow s}(h - j + 1, \tilde{n}_j)$  moves. Similarly, at phase **RightAccumulate**, a **right-move** of block  $B_j$  from peg  $j$  to peg  $d$  using  $h - j + 1$  available pegs is performed, requiring  $G_{j \rightarrow d}(h - j + 1, \tilde{n}_j)$  moves.

Consider now an index  $j \in [s, h' - 1]$ . At phase **RightSpread**, a **far-move** of block  $B_j$  from peg  $s$  to peg  $s + h - j$  using  $h - j + 1$  available pegs is performed, requiring  $F(h - j + 1, \tilde{n}_j)$  moves. At phase **LeftAccumulate**, a **left-move** of block  $B_j$  from peg  $s + h - j$  to peg  $d$  using  $h - j + 1$  available pegs is performed, requiring  $G_{d \rightarrow s+h-j}(h - j + 1, \tilde{n}_j)$  moves.

The remainder  $B(h')$  of blocks is moved in phase **MoveRemainder**, using a **far-move** from peg  $s$  to peg  $d$ , which requires  $F(h - h' + 1, n(h'))$  moves.

The discussion above implies

**Lemma 5.5**

$$\begin{aligned}
G_{s \rightarrow d}(h, n) &= \sum_{j=1}^{s-1} [G_{j \rightarrow s}(h-j+1, \tilde{n}_j) + G_{j \rightarrow d}(h-j+1, \tilde{n}_j)] \\
&\quad + \sum_{j=s}^{h'-1} [F(h-j+1, \tilde{n}_j) + G_{d \rightarrow s+h-j}(h-j+1, \tilde{n}_j)] \\
&\quad + F(h-h'+1, n(h')).
\end{aligned}$$

**5.4.2 Initial steps in the analysis of FarthestMove**

In this section we analyze the algorithm **FarthestMove** for moving a block  $B$  from peg 1 to peg  $h$  in  $\text{Path}_h$ ,  $h \geq 3$ , using the set  $A = [1, h]$  of available pegs.

First, observe that the last block  $B_{h-1}$ , namely disk  $B_{\max}$ , performs  $h-1$  moves.

Consider an index  $j \in [h-2]$ . At each of the phases **Spread**, **Reverse**, and **Accumulate**, a **far-move** of block  $B_j$  with  $h-j+1$  free pegs is performed, requiring a total of  $3F(h-j+1, \tilde{n}_j)$  moves. Also,  $j-1$  **shifts** of block  $B_j$  with  $h-j+1$  free pegs are performed at phase **Reverse**, requiring altogether  $(j-1)G_{1 \rightarrow 2}(h-j+1, \tilde{n}_j)$  moves.

For  $1 \leq i \leq h-1$ , denote by  $F(h, n)|_{n(i)}$  the number of moves of the  $n(i)$  largest disks in the course of performing the algorithm **FarthestMove**. The explanation in the preceding paragraph yields

**Lemma 5.6** *For any integers  $h \geq 2$ ,  $n \geq 1$ , and  $1 \leq i \leq h-1$ ,*

- $F(h, n)|_{n(i)} = \sum_{j=i}^{h-2} [3F(h-j+1, \tilde{n}_j) + (j-1)G_{1 \rightarrow 2}(h-j+1, \tilde{n}_j)] + h-1.$
- $F(h, n) = \sum_{j=1}^{i-1} [3F(h-j+1, \tilde{n}_j) + (j-1)G_{1 \rightarrow 2}(h-j+1, \tilde{n}_j)] + F(h, n)|_{n(i)}.$

For the subsequent lemmas we put  $m = h-k+1$ , for  $1 \leq k \leq h-1$ .

**Lemma 5.7** *For any integers  $h \geq 2$ ,  $n \geq 1$ , and  $1 \leq k \leq h-1$ ,*

$$F(h, n)|_{n(k)} = F(m, n(k)) + (k-1) \sum_{j=k}^{h-1} G_{1 \rightarrow 2}(h-j+1, \tilde{n}_j).$$

**Proof:** By Lemma 5.6 in the particular case  $i = 1$ ,

$$\begin{aligned} F(m, n(k)) &= \sum_{j=1}^{m-2} [3F(m-j+1, \tilde{n}_j(m, n(k))) \\ &\quad + (j-1)G_{1 \rightarrow 2}(m-j+1, \tilde{n}_j(m, n(k)))] + h - k. \end{aligned}$$

By Lemma 5.3, we obtain

$$\begin{aligned} F(m, n(k)) &= \sum_{j=1}^{m-2} [3F(m-j+1, \tilde{n}_{j+k-1}) \\ &\quad + (j-1)G_{1 \rightarrow 2}(m-j+1, \tilde{n}_{j+k-1})] + h - k \\ &= \sum_{j=k}^{h-2} [3F(h-j+1, \tilde{n}_j) + (j-k)G_{1 \rightarrow 2}(h-j+1, \tilde{n}_j)] + h - k. \end{aligned} \tag{7}$$

Observe that  $G_{1 \rightarrow 2}(2, \tilde{n}_{h-1}) = 1$ . Thus, by Lemma 5.6, the right-hand side of (7) reduces to

$$F(h, n)|_{n(k)} - (k-1) \sum_{j=k}^{h-1} G_{1 \rightarrow 2}(h-j+1, \tilde{n}_j),$$

and we are done.  $\blacksquare$

Lemma 5.6 and Lemma 5.7 imply

**Corollary 5.8** *For any integers  $h \geq 2, n \geq 1$ , and  $1 \leq k \leq h-1$ ,*

$$\begin{aligned} F(h, n) &= \sum_{j=1}^{k-1} [3F(h-j+1, \tilde{n}_j) + (j-1)G_{1 \rightarrow 2}(h-j+1, \tilde{n}_j)] \\ &\quad + F(m, n(k)) + (k-1) \sum_{j=k}^{h-1} G_{1 \rightarrow 2}(h-j+1, \tilde{n}_j). \end{aligned}$$

#### 5.4.3 Moving from one End to the Other is the most Costly

The following statement shows that `GeneralMove` requires the maximal number of moves when the source and destination pegs are at the extreme ends of the set  $A$ .

**Proposition 5.9** *For integers  $h \geq 3, n \geq 1, s, d$ , such that  $1 \leq s < d \leq h$  and  $d-s+1 < h$ ,*

$$G_{s \rightarrow d}(h, n) < F(h, n).$$

**Proof:** Denote  $h'' = h - h' + 1$ . The proof is by induction on  $n$ , for all values of  $h \geq 3$ . For  $n = 1$ , we have  $\tilde{n}_j = 0$  for each  $1 \leq j \leq h - 2$ . Hence by Lemma 5.5,

$$G_{s \rightarrow d}(h, 1) = F(h'', 1) = h'' - 1 = d - s < h - 1 = F(h, 1).$$

We assume that the statement holds for less than  $n$  disks and all  $h \geq 3$ , and prove it for  $n$  disks and all  $h \geq 3$ . Observe that

$$1 < s + h - d = h' \leq h - 1.$$

By Lemma 5.2, for each  $1 \leq j \leq h' - 1$ , we have  $\tilde{n}_j < n$ . Thus, by Lemma 5.5 and the induction hypothesis,

$$\begin{aligned} G_{s \rightarrow d}(h, n) &\leq \sum_{j=1}^{h'-1} 2F(h - j + 1, \tilde{n}_j) + F(h'', n(h')) \\ &< \sum_{j=1}^{h'-1} 2F(h - j + 1, \tilde{n}_j) + F(h'', n(h')) + (h' - 1). \end{aligned}$$

Since  $h' \leq h - 1$  and  $\tilde{n}_{h-1} = 1$ :

$$\sum_{j=h'}^{h-1} G_{1 \rightarrow 2}(h - j + 1, \tilde{n}_j) \geq G_{1 \rightarrow 2}(2, \tilde{n}_{h-1}) = 1.$$

Thus by Corollary 5.8,

$$\begin{aligned} F(h, n) &\geq \sum_{j=1}^{h'-1} 3F(h - j + 1, \tilde{n}_j) + F(h'', n(h')) + (h' - 1) \sum_{j=h'}^{h-1} G_{1 \rightarrow 2}(h - j + 1, \tilde{n}_j) \\ &\geq \sum_{j=1}^{h'-1} 2F(h - j + 1, \tilde{n}_j) + F(h'', n(h')) + (h' - 1) \\ &> G_{s \rightarrow d}(h, n). \quad \blacksquare \end{aligned}$$

## 5.5 Proof of Theorem 3.1

### 5.5.1 Auxiliary statements

**Lemma 5.10** For any integers  $h \geq 4$  and  $n \geq 1$ ,

$$G_{1 \rightarrow 2}(h, n) \leq \frac{2}{3}F(h, n) - 1.$$

**Proof:** By Lemma 5.5,

$$G_{1 \rightarrow 2}(h, n) = \sum_{j=1}^{h-2} [F(h-j+1, \tilde{n}_j) + G_{2 \rightarrow h-j+1}(h-j+1, \tilde{n}_j)] + F(2, n(h-1)).$$

Note that  $F(2, n(h-1)) = 1$ . Thus by Proposition 5.9,

$$G_{1 \rightarrow 2}(h, n) \leq \sum_{j=1}^{h-2} 2F(h-j+1, \tilde{n}_j) + 1.$$

Note that  $h-1 \geq 3$ . By Corollary 5.8 in the particular case  $k = h-1$ ,

$$\begin{aligned} F(h, n) &\geq \sum_{j=1}^{h-2} 3F(h-j+1, \tilde{n}_j) + F(2, n(h-1)) + (h-2)G_{1 \rightarrow 2}(2, \tilde{n}_{h-1}) \\ &\geq \sum_{j=1}^{h-2} 3F(h-j+1, \tilde{n}_j) + 1 + h-2 \\ &\geq \sum_{j=1}^{h-2} 3F(h-j+1, \tilde{n}_j) + 3. \end{aligned}$$

Altogether,

$$G_{1 \rightarrow 2}(h, n) \leq \sum_{j=1}^{h-2} 2F(h-j+1, \tilde{n}_j) + 1 \leq \frac{2}{3}F(h, n) - 1.$$

■

**Lemma 5.11** *For any integers  $h \geq 4$  and  $n \geq 2$ ,*

$$\sum_{j=1}^{h-1} G_{1 \rightarrow 2}(h-j+1, \tilde{n}_j) \leq \frac{2}{9}F(h, n).$$

**Proof:** By Lemma 5.6,

$$F(h, n) = \sum_{j=1}^{h-2} [3F(h-j+1, \tilde{n}_j) + (j-1)G_{1 \rightarrow 2}(h-j+1, \tilde{n}_j)] + h-1.$$

Note that  $\frac{2}{9}(h-1) \geq \frac{2}{3}$ . Hence,

$$\frac{2}{9}F(h, n) \geq \frac{2}{3} \left( \sum_{j=1}^{h-2} F(h-j+1, \tilde{n}_j) + 1 \right) = \frac{2}{3} \sum_{j=1}^{h-1} F(h-j+1, \tilde{n}_j). \quad (8)$$



We claim that

$$G_{1 \rightarrow 2}(h - j + 1, \tilde{n}_j) \leq \frac{2}{3}F(h - j + 1, \tilde{n}_j), \quad 1 \leq j \leq h - 3. \quad (9)$$

To this end, note that by Lemma 5.10 this clearly holds if  $\tilde{n}_j \geq 1$ . Otherwise  $\tilde{n}_j = 0$ , so

$$G_{1 \rightarrow 2}(h - j + 1, \tilde{n}_j) = 0 = \frac{2}{3}F(h - j + 1, \tilde{n}_j).$$

Since  $n \geq 2$  and  $\tilde{n}_{h-1} = 1$ , there exists a number  $j$  with  $1 \leq j \leq h - 2$  such that  $\tilde{n}_j \geq 1$ .

The analysis splits into two cases.

*Case 1:*  $\tilde{n}_{h-2} \geq 1$ .

By (9),

$$\sum_{j=1}^{h-3} G_{1 \rightarrow 2}(h - j + 1, \tilde{n}_j) \leq \frac{2}{3} \sum_{j=1}^{h-3} F(h - j + 1, \tilde{n}_j).$$

Note that  $F(3, \tilde{n}_{h-2}) \geq 2$ . Thus, by Lemma 5.4,

$$G_{1 \rightarrow 2}(3, \tilde{n}_{h-2}) = \frac{1}{2}F(3, \tilde{n}_{h-2}) \leq \frac{2}{3}F(3, \tilde{n}_{h-2}) - \frac{1}{3}.$$

Altogether, we have

$$\begin{aligned} \sum_{j=1}^{h-1} G_{1 \rightarrow 2}(h - j + 1, \tilde{n}_j) &= \sum_{j=1}^{h-3} G_{1 \rightarrow 2}(h - j + 1, \tilde{n}_j) + G_{1 \rightarrow 2}(3, \tilde{n}_{h-2}) + 1 \\ &\leq \frac{2}{3} \sum_{j=1}^{h-3} F(h - j + 1, \tilde{n}_j) + \frac{2}{3}F(3, \tilde{n}_{h-2}) + \frac{2}{3} \\ &= \frac{2}{3} \sum_{j=1}^{h-1} F(h - j + 1, \tilde{n}_j). \end{aligned} \quad (10)$$

By (8), the right-hand side of (10) is no greater than  $\frac{2}{9}F(h, n)$ , as required.

*Case 2:*  $\tilde{n}_i \geq 1$  for some  $i$  where  $1 \leq i \leq h - 3$ .

By (9) and Lemma 5.4,

$$\sum_{j \in [h-2] \setminus i} G_{1 \rightarrow 2}(h - j + 1, \tilde{n}_j) \leq \frac{2}{3} \sum_{j \in [h-2] \setminus i} F(h - j + 1, \tilde{n}_j).$$

By Lemma 5.10,

$$G_{1 \rightarrow 2}(h - i + 1, \tilde{n}_i) \leq \frac{2}{3}F(h - i + 1, \tilde{n}_i) - 1.$$

Altogether, we have

$$\begin{aligned} \sum_{j=1}^{h-1} G_{1 \rightarrow 2}(h-j+1, \tilde{n}_j) &= \sum_{j \in [h-2] \setminus \{i\}} G_{1 \rightarrow 2}(h-j+1, \tilde{n}_j) + G_{1 \rightarrow 2}(h-i+1, \tilde{n}_i) + 1 \\ &\leq \frac{2}{3} \sum_{j=1}^{h-2} F(h-j+1, \tilde{n}_j). \end{aligned} \quad (11)$$

By (8), the right-hand side of (11) is strictly less than  $\frac{2}{9}F(h, n)$ , and we are done.  $\blacksquare$

**Lemma 5.12** *For any integers  $h \geq 5$  and  $n \geq h$ ,*

$$F(h, n) \leq 3F(h, \tilde{n}_1) + \frac{11}{9}F(h-1, n - \tilde{n}_1).$$

**Proof:** By Corollary 5.8 in the particular case  $k = 2$ ,

$$F(h, n) = 3F(h, \tilde{n}_1) + F(h-1, n - \tilde{n}_1) + \sum_{j=2}^{h-1} G_{1 \rightarrow 2}(h-j+1, \tilde{n}_j).$$

We have  $n \geq h \geq 5$ . Thus,  $h-1 \geq 4$ , and by Lemma 5.1 we have  $n - \tilde{n}_1 \geq 2$ . Applying Lemma 5.11 with  $h-1$  and  $n - \tilde{n}_1$  instead of  $h$  and  $n$ , respectively, we get

$$\sum_{j=1}^{h-2} G_{1 \rightarrow 2}(h-j, \tilde{n}_j(h-1, n - \tilde{n}_1)) \leq \frac{2}{9}F(h-1, n - \tilde{n}_1). \quad (12)$$

By Lemma 5.3 in the particular case  $i = 2$ , for each  $1 \leq j \leq h-2$ ,

$$\tilde{n}_{j+1}(h, n) = \tilde{n}_j(h-1, n - \tilde{n}_1).$$

Consequently,

$$\sum_{j=2}^{h-1} G_{1 \rightarrow 2}(h-j+1, \tilde{n}_j) = \sum_{j=1}^{h-2} G_{1 \rightarrow 2}(h-j, \tilde{n}_j(h-1, n - \tilde{n}_1)) \leq \frac{2}{9}F(h-1, n - \tilde{n}_1),$$

which provides the required result.

**Lemma 5.13** *For any integers  $h \geq 5$  and  $1 \leq n < h$ ,*

$$F(h, n) \leq \tilde{U}(h, n) = \tilde{C}_h \cdot n^{\alpha_h} \cdot 3^{\theta_h \cdot n^{\frac{1}{h-2}}} < U(h, n),$$

where  $\theta_h$  and  $\alpha_h$  are as in Theorem 3.1, and  $\tilde{C}_h = \frac{h-2}{\theta_h}$ .

**Proof:** First note that  $\tilde{U}(h, n) < U(h, n)$ , so it remains to prove that  $F(h, n) \leq \tilde{U}(h, n)$ . Since  $h \geq 5$  and  $n < h$ , Lemmas 5.1 and 5.2 imply that  $\tilde{n}_i = 1$  for each  $i \in [n-1] \cup \{h-1\}$ . Thus by Lemma 5.6,

$$\begin{aligned} F(h, n) &= \sum_{j=1}^{h-2} [3F(h-j+1, \tilde{n}_j) + (j-1)G_{1 \rightarrow 2}(h-j+1, \tilde{n}_j)] + h-1 \\ &= \sum_{j=1}^{n-1} [3F(h-j+1, 1) + (j-1)G_{1 \rightarrow 2}(h-j+1, 1)] + h-1 \\ &= \sum_{j=1}^{n-1} 3(h-j) + (j-1) + h-1 = n(3h-n) - 2h. \end{aligned}$$

It is easy to verify that for  $h \geq 5$  and  $n < h$ ,

$$n(3h-n) - 2h \leq 3n(h-2).$$

Note that  $\theta_h n^{\frac{1}{h-2}} \geq 1$  for  $h \geq 3$  and  $n \geq 1$ . For  $x \geq 1$ , we have  $x \leq 3^{x-1}$ . Therefore,

$$3\theta_h n^{\frac{1}{h-2}} \leq 3^{\theta_h n^{\frac{1}{h-2}}},$$

and consequently,

$$3n(h-2) \leq \frac{h-2}{\theta_h} n^{\alpha_h} \cdot 3^{\theta_h n^{\frac{1}{h-2}}}.$$

Altogether,

$$F(h, n) \leq 3n(h-2) \leq \frac{h-2}{\theta_h} n^{\alpha_h} \cdot 3^{\theta_h n^{\frac{1}{h-2}}} = \tilde{U}(h, n).$$

■

### 5.5.2 Conclusion of the Proof

The proof is by double induction on  $h \geq 3$  and  $n$ .

For  $h = 3$ , the algorithm works exactly as does the algorithm of [27] for the 3-in-a-row graph. Therefore, the number  $F(3, n)$  of moves required by this algorithm for  $n$  disks is  $3^n - 1$ . The substitution  $h = 3$  in the upper bound  $U(h, n)$  suggested by the proposition yields:

$$U(3, n) = C_h n^{\alpha_h} \cdot 3^{\theta_h n^{1/\frac{1}{h-2}}} = 1 \cdot n^0 \cdot 3^{1 \cdot n} = 3^n > F(3, n).$$

For  $h = 4$ , the algorithm works exactly as does the algorithm FourMove of Section 4.2 for moving  $n$  disks between the two farthest pegs in  $\text{Path}_4$ . Therefore, as shown in Section

4.2,  $F(4, n)$  is bounded above by  $1.6\sqrt{n} \cdot 3^{\sqrt{2n}}$ . The substitution  $h = 4$  in the upper bound  $U(h, n)$  suggested by the proposition yields:

$$U(4, n) = C_h n^{\alpha_h} \cdot 3^{\theta_h n^{\frac{1}{h-2}}} = c\sqrt{2n} \cdot 3^{\sqrt{2n}} > 1.6\sqrt{n} \cdot 3^{\sqrt{2n}} \geq F(4, n).$$

For  $h \geq 5$  and  $n < h$ , Lemma 5.13 implies that  $F(h, n) < U(h, n)$ .

We assume that for arbitrary fixed  $h \geq 5$  and  $n \geq h$ ,  $F(h', n') < U(h', n')$  holds for all  $(h', n')$  with either  $h' < h$  or both  $h' = h$  and  $n' < n$ , and prove it for  $(h, n)$ .

Let  $m = m_1 = \text{Remainder}(h, n)$ . Then  $\tilde{n}_1 = n - m$ . By Lemma 5.12,

$$F(h, n) \leq 3F(h, n - m) + \frac{11}{9}F(h - 1, m).$$

The analysis splits into two cases.

**Case 1:**  $n \leq \frac{(h-2)^{h-2}}{(h-2)!}$ .

In this case, we have

$$n \leq \frac{((h-2)!)^{\alpha_h}}{(h-3)!} n^{\alpha_h}, \quad (13)$$

and so,

$$m = \text{Remainder}(h, n) = \min \left\{ n, \left\lceil \frac{((h-2)!)^{\alpha_h}}{(h-3)!} n^{\alpha_h} \right\rceil \right\} = n.$$

It follows that  $F(h, n) \leq \frac{11}{9}F(h-1, n)$ . By the induction hypothesis,

$$F(h, n) < \frac{11}{9}U(h-1, n) = \frac{11}{9}C_{h-1} \cdot n^{\alpha_{h-1}} \cdot 3^{\theta_{h-1} \cdot n^{\frac{1}{h-3}}}. \quad (14)$$

By (13), we have

$$\theta_{h-1} \cdot n^{\frac{1}{h-3}} \leq \theta_{h-1} \left( \frac{((h-2)!)^{\alpha_h}}{(h-3)!} n^{\alpha_h} \right)^{\frac{1}{h-3}} = \theta_{h-1} \left( \frac{\theta_h^{h-3}}{\theta_{h-1}^{h-3}} n^{\frac{h-3}{h-2}} \right)^{\frac{1}{h-3}} = \theta_h \cdot n^{\frac{1}{h-2}}. \quad (15)$$

Substituting (15) in (14), we obtain

$$F(h, n) < \frac{11}{9}C_{h-1} \cdot n^{\alpha_{h-1}} \cdot 3^{\theta_h \cdot n^{\frac{1}{h-2}}}. \quad (16)$$

It is easy to verify that  $\frac{11}{9}C_{h-1} < C_h$  and  $\alpha_{h-1} < \alpha_h$ . Thus we find that:

$$F(h, n) < C_h \cdot n^{\alpha_h} \cdot 3^{\theta_h \cdot n^{\frac{1}{h-2}}}.$$

**Case 2:**  $n > \frac{(h-2)^{h-2}}{(h-2)!}$ .

In this case, we have

$$n > \frac{((h-2)!)^{\alpha_h}}{(h-3)!} n^{\alpha_h},$$

and so,

$$m = \text{Remainder}(h, n) = \min \left\{ n, \left\lceil \frac{((h-2)!)^{\alpha_h}}{(h-3)!} n^{\alpha_h} \right\rceil \right\} = \left\lceil \frac{((h-2)!)^{\alpha_h}}{(h-3)!} n^{\alpha_h} \right\rceil \leq n.$$

By the induction hypothesis,

$$F(h, n) < 3C_h(n-m)^{\alpha_h} 3^{\theta_h \cdot (n-m)^{\frac{1}{h-2}}} + \frac{11}{9} C_{h-1} \cdot m^{\alpha_{h-1}} \cdot 3^{\theta_{h-1} \cdot m^{\frac{1}{h-3}}}. \quad (17)$$

Observe that

$$(n-m)^{\alpha_h} = n^{\alpha_h} \left(1 - \frac{m}{n}\right)^{\alpha_h} < n^{\alpha_h} \left(1 - \frac{\alpha_h m}{n}\right). \quad (18)$$

Similarly, we have

$$\theta_h(n-m)^{\frac{1}{h-2}} = \theta_h n^{\frac{1}{h-2}} \left(1 - \frac{m}{n}\right)^{\frac{1}{h-2}} < \theta_h n^{\frac{1}{h-2}} \left(1 - \frac{\theta_h^{h-3} \cdot n^{\frac{h-3}{h-2}}}{(h-2)n\theta_{h-1}^{h-3}}\right) = \theta_h n^{\frac{1}{h-2}} - 1 \quad (19)$$

and

$$\begin{aligned} \theta_{h-1} \cdot m^{\frac{1}{h-3}} &\leq \theta_{h-1} \left( \frac{\theta_h^{h-3}}{\theta_{h-1}^{h-3}} \cdot n^{\frac{h-3}{h-2}} + 1 \right)^{\frac{1}{h-3}} = \theta_h n^{\frac{1}{h-2}} \left( 1 + \frac{\theta_h^{h-3}}{\theta_{h-1}^{h-3}} n^{-\frac{h-3}{h-2}} \right)^{\frac{1}{h-3}} \\ &< \theta_h n^{\frac{1}{h-2}} + \frac{(h-3)! \cdot \theta_h^2}{(h-3)(h-2)!} n^{-\frac{h-4}{h-2}} = \theta_h n^{\frac{1}{h-2}} + \frac{\theta_h^2}{(h-3)(h-2)} n^{-\frac{h-4}{h-2}}. \end{aligned}$$

It is easy to verify that  $\vartheta(h, n) = \frac{\theta_h^2}{(h-3)(h-2)} n^{-\frac{h-4}{h-2}}$  is monotone decreasing with  $h$  and  $n$  in the range  $n \geq h \geq 5$ . Hence for  $n \geq h \geq 5$ , we have

$$\vartheta(h, n) \leq \vartheta(5, 5) = \left(\frac{1}{30}\right)^{1/3}.$$

Put  $c^* = \left(\frac{1}{30}\right)^{1/3}$ . Now

$$\theta_{h-1} \cdot m^{\frac{1}{h-3}} < \theta_h \cdot n^{\frac{1}{h-2}} + c^*. \quad (20)$$

Substituting (18), (19) and (20) in (17), we obtain

$$\begin{aligned} F(h, n) &< 3C_h n^{\alpha_h} \left(1 - \frac{\alpha_h m}{n}\right) 3^{\theta_h \cdot n^{\frac{1}{h-2}} - 1} + \frac{11}{9} C_{h-1} \cdot m^{\alpha_{h-1}} \cdot 3^{\theta_{h-1} \cdot m^{\frac{1}{h-3}} + c^*} \\ &= C_h n^{\alpha_h} \left(1 - \frac{\alpha_h m}{n}\right) 3^{\theta_h \cdot n^{\frac{1}{h-2}}} + \frac{11}{9} 3^{c^*} C_{h-1} \cdot m^{\alpha_{h-1}} \cdot 3^{\theta_{h-1} \cdot m^{\frac{1}{h-3}}} \\ &= \left(C_h n^{\alpha_h} - \frac{C_h \cdot n^{\alpha_h} \cdot \alpha_h \cdot m}{n} + \delta \cdot C_{h-1} \cdot m^{\alpha_{h-1}}\right) 3^{\theta_h \cdot n^{\frac{1}{h-2}}}. \end{aligned} \quad (21)$$

The second and third terms on the right-hand side of (21) may be omitted since:

$$\begin{aligned}
\delta C_{h-1} \cdot m^{\alpha_{h-1}} - \frac{C_h \cdot n^{\alpha_h} \cdot \alpha_h \cdot m}{n} &= m^{\alpha_{h-1}} \left( \delta \frac{(h-3)\delta^{h-4}}{\theta_{h-1}} - \frac{(h-2)\delta^{h-3}}{n \cdot \theta_h} \cdot n^{\frac{h-3}{h-2}} \cdot \frac{h-3}{h-2} \cdot m^{\frac{1}{h-3}} \right) \\
&= m^{\alpha_{h-1}} (h-3)\delta^{h-3} \left( \frac{1}{\theta_{h-1}} - \frac{n^{\frac{-1}{h-2}} \cdot m^{\frac{1}{h-3}}}{\theta_h} \right) \\
&\leq m^{\alpha_{h-1}} (h-3)\delta^{h-3} \left( \frac{1}{\theta_{h-1}} - n^{\frac{-1}{h-2}} \left( \frac{\theta_h^{h-3}}{\theta_{h-1}^{h-3}} \cdot n^{\frac{h-3}{h-2}} \right)^{\frac{1}{h-3}} \right) \\
&= 0.
\end{aligned}$$

Thus we conclude that:

$$F(h, n) < C_h n^{\alpha_h} \cdot 3^{\theta_h \cdot n^{\frac{1}{h-2}}}.$$

■

## References

- [1] J.-P. Allouche, D. Astoorian, J. Randall, and J. Shallit. Morphisms, squarefree strings, and the Tower of Hanoi puzzle. *Amer. Math. Monthly*, 101:651–658, 1994.
- [2] J.-P. Allouche and A. Sapir. Restricted Towers of Hanoi and morphisms. *LNCS*, 3572:1–10, 2005.
- [3] M. D. Atkinson. The cyclic Towers of Hanoi. *Inform. Process. Lett.*, 13:118–119, 1981.
- [4] D. Azriel and D. Berend. On a question of Leiss regarding the Hanoi Tower problem. *Theoretical Computer Science*, 369:377–383, 2006.
- [5] D. Azriel, N. Solomon, and S. Solomon. On an infinite family of solvable Hanoi graphs. *Trans. on Algorithms*, 5(1), 2008.
- [6] D. Berend and A. Sapir. The Cyclic multi-peg Tower of Hanoi. *Trans. on Algorithms*, 2(3):297–317, 2006.
- [7] D. Berend and A. Sapir. The diameter of Hanoi graphs. *Inform. Process. Lett.*, 98:79–85, 2006.

- [8] X. Chen and J. Shen. On the Frame-Stewart conjecture about the Towers of Hanoi. *SIAM J. on Computing*, 33(3):584–589, 2004.
- [9] Y. Dinitz and S. Solomon. Optimal algorithms for Tower of Hanoi problems with relaxed placement rules. *Proc. of ISSAC06*, pages 36–47, 2006.
- [10] Y. Dinitz and S. Solomon. On Optimal solutions for the Bottleneck Tower of Hanoi problem. *Proc. of SOFSEM07*, pages 248–259, 2007.
- [11] Y. Dinitz and S. Solomon. Optimality of an algorithm solving the Bottleneck Tower of Hanoi problem. *Trans. on Algorithms*, 4(3):1–9, 2008.
- [12] H. E. Dudeney. *“The Canterbury Puzzles (and Other Curious Problems)”*. E. P. Dutton, New York, 1908.
- [13] M. C. Er. The Cyclic Towers of Hanoi: a representation approach. *Comput. J.*, 27(2):171–175, 1984.
- [14] M. C. Er. The complexity of the generalised Cyclic Towers of Hanoi. *J. Algorithms*, 6:351–358, 1985.
- [15] J. S. Frame. Solution to advanced problem 3918. *Amer. Math. Monthly*, 48:216–217, 1941.
- [16] A. M. Hinz. Pascal’s triangle and the Tower of Hanoi. *Amer. Math. Monthly*, 99:538–544, 1992.
- [17] S. Klavžar, U. Milutinović, and C. Petr. On the Frame-Stewart algorithm for the multi-peg Tower of Hanoi problem. *Discrete Applied Math.*, 120(1-3):141–157, 2002.
- [18] S. Klavžar, U. Milutinović, and C. Petr. Hanoi graphs and some classical numbers. *Expo. Math.*, 23(4):371–378, 2005.
- [19] E. L. Leiss. Finite Hanoi problems: how many discs can be handled? *Congr. Numer.*, 44(1):221–229, 1984.
- [20] É. Lucas. *“Récréations Mathématiques”*, volume III. Gauthier-Villars, Paris, 1893.
- [21] W. F. Lunnon and P. K. Stockmeyer. New Variations on the Tower of Hanoi.
- [22] S. Minsker. The Little Towers of Antwerpen problem. *Inform. Process. Lett.*, 94(5):197–201, 2005.

- [23] S. Minsker. The Linear Twin Towers of Hanoi problem. *ACM SIGCSE Bull.*, 39(4):37–40, 2007.
- [24] S. Minsker. Another brief recursion excursion to Hanoi. *ACM SIGCSE Bull.*, 40(4):35–37, 2008.
- [25] S. Minsker. The classical/linear Hanoi hybrid problem: regular configurations. *ACM SIGCSE Bull.*, 41(4):57–61, 2009.
- [26] A. Sapir. The Tower of Hanoi with forbidden moves. *Comput. J.*, 47(1):20–24, 2004.
- [27] R. S. Scorer, P. M. Grundy, and C. A. B. Smith. Some binary games. *Math. Gazette*, 280:96–103, 1944.
- [28] B. M. Stewart. Advanced problem 3918. *Amer. Math. Monthly*, 46:363, 1939.
- [29] B. M. Stewart. Solution to advanced problem 3918. *Amer. Math. Monthly*, 48:217–219, 1941.
- [30] P. K. Stockmeyer. Variations on the Four-Post Tower of Hanoi puzzle. *Congr. Numer.*, 102:3–12, 1994.
- [31] P. K. Stockmeyer. The average distance between nodes in the Cyclic Tower of Hanoi digraph. *Graph Theory, Combinatorics, Algorithms, and Applications*, 1996.
- [32] M. Szegedy. In how many steps the  $k$  peg version of the Towers of Hanoi game can be solved? *Lect. Notes in Comput. Sci.*, 1563:356–361, 1999.